

Spectrum of Navier p -biharmonic problem with sign-changing weight *

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Abstract

In this paper, we consider the following eigenvalue problem

$$\begin{cases} (|u''|^{p-2}u'')'' = \lambda m(x)|u|^{p-2}u, & x \in (0, 1), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases}$$

where $1 < p < +\infty$, λ is a real parameter and m is sign-changing weight. We prove there exists a unique sequence of eigenvalues for above problem. Each eigenvalue is simple and continuous with respect to p , the k -th eigenfunction, corresponding to the k -th positive or negative eigenvalue, has exactly $k - 1$ generalized simple zeros in $(0, 1)$.

Keywords: Spectrum; p -biharmonic operator; Sign-changing weight

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1 Introduction

It is well known that fourth-order elliptic problems arise in many applications, such as Micro Electro Mechanical systems, thin film theory, surface diffusion on solids, interface dynamics, flow in Hele-Shaw cells, phase field models of multi-phase systems and the deformation of an elastic beam, see, for example, [16, 28] and the references therein. Thus, there are many papers concerning the existence and multiplicity of positive solutions and sign-changing solutions addressed by using different methods, such as those of topology degree theory, critical point theory, the fixed point theorem in cones and bifurcation techniques [2, 4, 23, 24, 26, 27, 33]. We also have known that the p -Laplacian with $p \neq 2$ arises in, for example, the study of non-Newtonian fluids ($p > 2$ for dilatant fluids and $p < 2$ for pseudoplastic fluids), in torsional creep problems ($p \geq 2$), as well as in glaciology ($p \in (1, 4/3]$) (see [20]). Problems with p -biharmonic operator attracted growing interest, and figures in a variety of applications, where this operator is used to control the nonlinear artificial viscosity or diffusion of non-Newtonian fluids (see [12]). Problem with sign-changing weight arises from the selection-migration model in population genetics. In this model, m changes sign corresponding to the fact that an allele A_1 holds an advantage over

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a rival allele A_2 at same points and is at a disadvantage at others; the parameter θ corresponds to the reciprocal of diffusion, for details, see [17].

Recently, Ma et al. [25] established the existence of the principal eigenvalues of the following linear indefinite weight problem

$$\begin{cases} u'''' = \lambda g(x)u, & x \in (0, 1), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases}$$

where $g : [0, 1] \rightarrow \mathbb{R}$ is a continuous sign-changing function. However, there is no any information on the high eigenvalue.

In [11], Drábek and Ôtani considered the following one dimensional eigenvalue problem

$$\begin{cases} (|u''|^{p-2}u'')'' = \lambda |u|^{p-2}u, & x \in (0, 1), \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}$$

They gave a complete characterization of the spectrum of this problem by using the abstract result of Idogawa and Ôtani [21]. Following, J. Benedikt [5, 6] established the similar results for the Dirichlet, Neumann and Robin p -biharmonic problem. In [13], the authors studied the spectrum of the following eigenvalue problem

$$\begin{cases} \Delta_p^2 u = \lambda \rho(x) |u|^{p-2}u, & x \in \Omega, \\ u \in W_0^{2,p}(\Omega), \end{cases}$$

where Ω is a bounded domain in \mathbb{R}^N with $N \geq 1$, $\Delta_p^2 u := \Delta(|\Delta u|^{p-2}\Delta u)$ is the p -bihaimonic operator, $\rho \in L^s(\Omega)$ is a unbounded weight function which can change its sign, with s satisfying the conditions

$$s \begin{cases} > \frac{N}{2p}, & \text{for } \frac{N}{p} \geq 2, \\ = 1, & \text{for } \frac{N}{p} < 2. \end{cases}$$

They showed that the above problem has at least two sequences of eigenvalues

$$0 < \lambda_1^+ \leq \lambda_2^+ \leq \dots < \lambda_k^+ \leq \dots, \lim_{k \rightarrow +\infty} \lambda_k^+ = +\infty,$$

$$0 > \lambda_1^- \geq \lambda_2^- \geq \dots > \lambda_k^- \geq \dots, \lim_{k \rightarrow +\infty} \lambda_k^- = -\infty.$$

However, no any information has been obtained on the simple and isolated properties of the above eigenvalues, even in the case of $N = 1$, which are very important in the study of the global bifurcation phenomena for p -bihaimonic problems, see [7, 11]. Moreover, we dot not see whether or not there are other eigenvalues different from the above.

Motivated by above papers, we study the following one dimensional p -bihaimonic eigenvalue problem with $1 < p < +\infty$

$$\begin{cases} \Delta_p^2 u = \lambda m(x) |u|^{p-2}u, & x \in (0, 1), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \quad (1.1)$$

where $m(x)$ is a sign-changing weight function, $\Delta_p^2 u := (|u''|^{p-2}u'')''$. Let $I := (0, 1)$ and

$$M(I) := \left\{ m \in C(\bar{I}) \mid \text{meas}\{x \in I \mid m(x) > 0\} > 0 \right\}.$$

We shall show that:

Theorem 1.1. *Let $m \in M(I)$. The eigenvalue problem (1.1) has two sequences of simple real eigenvalues*

$$0 < \lambda_1^+(p) < \lambda_2^+(p) < \cdots < \lambda_k^+(p) < \cdots, \lim_{k \rightarrow +\infty} \lambda_k^+(p) = +\infty,$$

$$0 > \lambda_1^-(p) > \lambda_2^-(p) > \cdots > \lambda_k^-(p) > \cdots, \lim_{k \rightarrow +\infty} \lambda_k^-(p) = -\infty$$

and no other eigenvalues. Moreover, for $k \in \mathbb{N}$ and $\nu \in \{+, -\}$, we have

1. *eigenfunction corresponding to eigenvalue $\lambda_k^\nu(p)$, has exactly $k-1$ generalized simple zeros;*
2. *$\lambda_k^\nu(p)$ is continuous with respect to p .*

The proof is based on variational method, monotone operator theory and convex analysis techniques. To the best of our knowledge, partial results of Theorem 1.1 are new even in the case of $m(x) \geq 0$ in \bar{I} or $p = 2$.

It is worth to remark that the sign-changing property of weight function m raises some essential difficulties. For example, the Strong Maximum Principle of [14, 18] can not be used directly in this paper and the operator $\Theta(u) := \int_I m(x)|u|^{p-2}u \, dx$ is not monotone any more. Another main difficulty which we need to overcome is raised by p -biharmonic operator, for example, Picone's identity method will not be holding for problem (1.1) and the Existence and Uniqueness Theorem of [32] is also invalid for problem (1.1), while they are very important in the proof of the simple properties of eigenvalues and the zeros of eigenfunctions, respectively.

The rest of this paper is arranged as follows. In Section 2, we recall/prove some preliminary results which will be used later. In Section 3, we prove the existence of the principal eigenvalues for problem (1.1) and study some properties of the principal eigenvalues. In the last Section, we establish the existence of the discrete eigenvalues for problem (1.1) and conclude the proof of Theorem 1.1.

2 Preliminaries

From now on, let X be $W_0^{1,p}(I) \cap W^{2,p}(I)$ with the norm $\|u\| := (\int_I |u''|^p \, dx)^{1/p}$ and X^* the dual space of X . For simplicity we write $u_n \rightharpoonup u$ and $u_n \rightarrow u$ to denote the weak convergence and strong convergence of sequence $\{u_n\}$ in X , respectively. Firstly, we recall the definition of weak solution.

Definition 2.1. $u \in X$ is called a weak solution of problem (1.1) if

$$\int_I |u''|^{p-2} u'' \phi'' \, dx = \lambda \int_I m(x) \varphi_p(u) \phi \, dx \quad (2.1)$$

for any $\phi \in X$, where $\varphi_p(u) = |u|^{p-2}u$.

For the regularity of weak solution, we have the following result.

Proposition 2.1. *Any weak solution $u \in X$ of problem (1.1) is also a classical solution of problem (1.1), i.e., $u \in C^2(I)$, $\varphi_p(u'') \in C^2(I)$ and $u''(0) = u''(1) = 0$.*

In order to prove Proposition 2.1, we need the following technical lemma.

Lemma 2.1. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function. For a given $x_0 \in \mathbb{R}$, if f is continuous in some neighborhood U of x_0 , differential in $U \setminus \{x_0\}$ and $\lim_{x \rightarrow x_0} f'(x)$ exists, then f is differential at x_0 and $f'(x_0) = \lim_{x \rightarrow x_0} f'(x)$.*

Proof. The conclusion is a direct corollary of Lagrange mean Theorem, we omit its proof here. ■

Proof of Proposition 2.1. Firstly, by the embedding theorem (see [14]), we have $X \hookrightarrow C^{1,\alpha}(\bar{I})$ with $\alpha = 1 - 1/p$, it follows that $u \in C^{1,\alpha}(\bar{I})$. According to Definition 2.1, we have

$$(\varphi_p(u''))'' = \lambda m(x) \varphi_p(u) \quad (2.2)$$

in I in the sense of distribution, i.e.,

$$(\varphi_p(u''))'' = \lambda m(x) \varphi_p(u) \text{ in } I \setminus I_0 \quad (2.3)$$

for some $I_0 \subset I$ which satisfies $\text{meas}\{I_0\} = 0$. Let $v := \varphi_p(u'')$. Then the equation (2.3) implies that $v \in C^2(I \setminus I_0)$. Furthermore, $u'' = \varphi_{p'}(v) \in C(I \setminus I_0)$ where $p' = p/(p-1)$. For fixed x_* , $x^* \in I \setminus I_0$, (2.3) follows that

$$v(x) = v(x^*) + \int_{x^*}^x \left(v'(x_*) + \int_{x_*}^t \lambda m(\tau) \varphi_p(u(\tau)) d\tau \right) dt, \quad x \in I \setminus I_0. \quad (2.4)$$

For any $x_0 \in I_0$, it is easy to see that (2.4) implies the existence of $\lim_{x \rightarrow x_0} v(x)$, hence $\lim_{x \rightarrow x_0} u''(x)$ exists. Lemma 2.1 and $u \in C^{1,\alpha}(\bar{I})$ follow that $u''(x_0) = \lim_{x \rightarrow x_0} u''(x)$. By the arbitrary property of x_0 , we get $u \in C^2(I)$.

Clearly, $u \in C^2(I)$ implies that $v \in C(I)$. For above x_* , (2.3) follows that

$$v'(x) = v'(x_*) + \int_{x_*}^x \lambda m(t) \varphi_p(u(t)) dt, \quad x \in I \setminus I_0. \quad (2.5)$$

It is obvious that (2.5) implies $\lim_{x \rightarrow x_0} v'(x)$ exists. Using Lemma 2.1, we get $v'(x_0) = \lim_{x \rightarrow x_0} v'(x)$, that is to say $v'(x)$ is continuous at x_0 . Clearly, (2.3) implies that $\lim_{x \rightarrow x_0} v''(x)$ exists. By Lemma 2.1 again, we have $v''(x_0) = \lim_{x \rightarrow x_0} v''(x)$. Therefore, $v \in C^2(I)$ since x_0 is arbitrary in I_0 .

It remains to show that $u''(0) = u''(1) = 0$. Let us choose $v \in C_0^2(\bar{I}) \subset X$. Substituting in (2.1) and integrating by parts we obtain

$$u''(1)v'(1) - u''(0)v'(0) = 0.$$

Since $v'(1)$ and $v'(0)$ are arbitrary, it follows from the last equality that $u''(0) = u''(1) = 0$. ■

Let u be a nontrivial weak solution of (1.1), we introduce the definition of the “simple” property of the zero of u .

Definition 2.2. Let u be a nontrivial weak solution of (1.1) and x_* be a zero of u . We call the zero x_* is generalized simple if $u''(x_*) = 0$ but $u'(x_*) \neq 0$ or $(\varphi_p(u''))' \neq 0$.

Consider the following two functional defined on X :

$$A(u) = \frac{1}{p} \int_I |u''|^p dx, \quad B(u) = \frac{1}{p} \int_I m(x) |u|^p dx.$$

We know that $A \in C^1(X, \mathbb{R})$ and the operator Δ_p^2 is the derivative operator of A in the weak sense. We denote $L = A' : X \rightarrow X^*$, then

$$\langle L(u), v \rangle = \int_I |u''|^{p-2} u'' v'' dx, \quad \forall u, v \in X, \quad (2.6)$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between X and X^* .

We have the following properties about the operator L .

Proposition 2.2. (i) $L : X \rightarrow X^*$ is a continuous, bounded and strictly monotone operator;

(ii) L is a mapping of type (S_+) , i.e., if $u_n \rightharpoonup u$ in X and $\overline{\lim}_{n \rightarrow +\infty} \langle L(u_n) - L(u), u_n - u \rangle \leq 0$, then $u_n \rightarrow u$ in X ;

(iii) $L : X \rightarrow X^*$ is a homeomorphism.

Proof. (i) It is obvious that L is continuous and bounded. For any $u, w \in X$ with $u \neq w$ in X . By Cauchy's inequality, we have

$$u'' w'' \leq |u''| |w''| \leq \frac{|u''|^2 + |w''|^2}{2}. \quad (2.7)$$

Using (2.7), we can easily obtain that

$$\int_I |u''|^p dx - \int_I |u''|^{p-2} u'' w'' dx \geq \int_I \frac{|u''|^{p-2}}{2} (|u''|^2 - |w''|^2) dx \quad (2.8)$$

and

$$\int_I |w''|^p dx - \int_I |w''|^{p-2} u'' w'' dx \geq \int_I \frac{|w''|^{p-2}}{2} (|w''|^2 - |u''|^2) dx. \quad (2.9)$$

By (2.6), (2.8) and (2.9), we obtain

$$\begin{aligned} \langle L(u) - L(w), u - w \rangle &= \langle L(u), u \rangle - \langle L(u), w \rangle - \langle L(w), u \rangle + \langle L(w), w \rangle \\ &= \left(\int_I |u''|^p dx - \int_I |u''|^{p-2} u'' w'' dx \right) - \left(\int_I |w''|^{p-2} u'' w'' dx - \int_I |w''|^p dx \right) \\ &\geq \int_I \frac{|u''|^{p-2}}{2} (|u''|^2 - |w''|^2) dx - \int_I \frac{|w''|^{p-2}}{2} (|u''|^2 - |w''|^2) dx \\ &\geq \int_I \frac{1}{2} (|u''|^{p-2} - |w''|^{p-2}) (|u''|^2 - |w''|^2) dx \geq 0, \end{aligned} \quad (2.10)$$

i.e., L is monotone. In fact, L is strictly monotone. Indeed, if $\langle L(u) - L(w), u - w \rangle = 0$, then we have

$$\int_I (|u''|^{p-2} - |w''|^{p-2}) (|u''|^2 - |w''|^2) dx = 0,$$

so $|u''| = |w''|$. Thus, we obtain

$$\langle L(u) - L(w), u - w \rangle = \langle L(u), u - w \rangle - \langle L(w), u - w \rangle = \int_I |u''|^{p-2} (u'' - w'')^2 dx = 0. \quad (2.11)$$

If $1 < p < 2$, (2.11) follows that $u'' \equiv w''$, which is a contradiction. If $p \geq 2$, (2.11) implies that $u'' \equiv w''$ which contradicts $u \neq w$ in X or $|u''| \equiv 0$. If $|u''| \equiv 0$, in view of $|u''| = |w''|$, we get $u'' \equiv w''$, which is a contradiction again. Therefore, $\langle L(u) - L(w), u - w \rangle > 0$. It follows that L is a strictly monotone operator on X .

(ii) From (i), if $u_n \rightharpoonup u$ and $\lim_{n \rightarrow +\infty} \langle L(u_n) - L(u), u_n - u \rangle \leq 0$, then $\lim_{n \rightarrow +\infty} \langle L(u_n) - L(u), u_n - u \rangle = 0$. In view of (2.10), u_n'' converges in measure to u'' in I , so we get a subsequence (which we still denote by u_n) satisfying $u_n''(x) \rightarrow u''(x)$, a.e. $x \in I$. By Fatou's Lemma we get

$$\liminf_{n \rightarrow +\infty} \int_I \frac{1}{p} |u_n''|^p dx \geq \int_I \frac{1}{p} |u''|^p dx. \quad (2.12)$$

From $u_n \rightharpoonup u$ we have $\lim_{n \rightarrow +\infty} \langle L(u_n), u_n - u \rangle = \lim_{n \rightarrow +\infty} \langle L(u_n) - L(u), u_n - u \rangle = 0$. On the other hand, by Young's inequality, we have

$$\begin{aligned} \langle L(u_n), u_n - u \rangle &= \int_I |u_n''|^p dx - \int_I |u_n''|^{p-2} u_n'' u'' dx \\ &\geq \int_I |u_n''|^p dx - \int_I |u_n''|^{p-1} |u''| dx \\ &\geq \int_I \frac{1}{p} |u_n''|^p dx - \int_I \frac{1}{p} |u''|^p dx. \end{aligned} \quad (2.13)$$

According to (2.12)–(2.13) we obtain

$$\lim_{n \rightarrow +\infty} \int_I \frac{1}{p} |u_n''|^p dx = \int_I \frac{1}{p} |u''|^p dx.$$

Using the method similar to [15], we have

$$\lim_{n \rightarrow +\infty} \int_I |u_n'' - u''|^p dx = 0.$$

Therefore, $u_n \rightarrow u$, i.e., L is of type (S_+) .

(iii) It is clear that L is an injection since L is a strictly monotone operator on X . Since

$$\lim_{\|u\| \rightarrow +\infty} \frac{\langle L(u), u \rangle}{\|u\|} = \lim_{\|u\| \rightarrow +\infty} \frac{\int_I |u''|^p dx}{\|u\|} = +\infty,$$

L is coercive, thus L is a surjection in view of Minty-Browder Theorem (see [34, Theorem 26A]). Hence L has an inverse mapping $L^{-1} : X^* \rightarrow X$. Therefore, the continuity of L^{-1} is sufficient to ensure L to be a homeomorphism.

If $f_n, f \in X^*$, $f_n \rightarrow f$, let $u_n = L^{-1}(f_n)$, $u = L^{-1}(f)$, then $L(u_n) = f_n$, $L(u) = f$. The coercive property of L implies that $\{u_n\}$ is bounded in X . Without loss of generality, we can assume that $u_n \rightharpoonup u$. Since $f_n \rightarrow f$, then

$$\lim_{n \rightarrow +\infty} \langle L(u_n) - L(u), u_n - u \rangle = \lim_{n \rightarrow +\infty} \langle f_n, u_n - u \rangle = 0.$$

Since L is of type (S_+) , $u_n \rightarrow u$, so L is continuous. ■

Remark 2.1. We would like to point out that the property (iii) of L will not be used later. However, this property is interesting and will be used in our following paper.

Remark 2.2. Let $a(x)$ be a non-negative continuous function defined on \bar{I} and

$$\mathcal{A}(u) = \frac{1}{p} \int_I (|u''|^p + a(x)|u|^p) dx.$$

Repeating the argument of Proposition 2.2, we can show that the derivative operator of \mathcal{A} which we denote by \mathcal{L} is also a strictly monotone operator.

Set $\mathbb{M} = \{u \in X \mid pB(u) = 1\}$ and

$$\Lambda_k = \{K \subset \mathbb{M} \mid K \text{ is symmetric, compact and } \gamma(K) \geq k\},$$

where $k \in \mathbb{N}$, $\gamma(K)$ is the genus of K . Corollary 3.7 and Corollary 3.8 of [13] imply that problem (1.1) has at least two sequences of eigenvalues

$$0 < \lambda_1^+ \leq \lambda_2^+ \leq \dots < \lambda_k^+ \leq \dots, \lim_{k \rightarrow +\infty} \lambda_k^+ = +\infty,$$

$$0 > \lambda_1^- \geq \lambda_2^- \geq \dots > \lambda_k^- \geq \dots, \lim_{k \rightarrow +\infty} \lambda_k^- = -\infty,$$

and $\lambda_k^+ = \inf_{K \in \Lambda_k} \max_{u \in K} pA(u)$ for any $k > 1$. Moreover,

$$\lambda_1^+ = \inf \left\{ \int_I |u''|^p dx \mid u \in X, \int_I m(x)|u|^p = 1 \right\}. \quad (2.14)$$

Clearly, 0 is not the eigenvalue of problem (1.1). From now on, we only consider the case of $\lambda > 0$. In the case of $\lambda < 0$, we consider a new sign-changing eigenvalue problem

$$\begin{cases} \Delta_p^2 u = \hat{\lambda} \hat{m}(x)u, & x \in I, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases}$$

where $\hat{\lambda} = -\lambda$, $\hat{m}(x) = -m(x)$. It is easy to check that

$$\hat{\lambda}_k^+ = -\lambda_k^-, \quad k \in \mathbb{N}.$$

Thus, we may use the results obtained in the case of $\lambda > 0$ to deduce the desired result.

3 The existence of the principal eigenvalues

In this section, we shall show that λ_1^+ is the principal eigenvalue of problem (1.1). Moreover, we shall also give some basic properties of λ_1^+ which will be used later.

Proposition 3.1. *The infimum λ_1^+ in (2.14) is achieved at some $\psi \in \mathbb{M}$ and λ_1^+ is the least positive eigenvalue of problem (1.1). Moreover, $\lambda_1^+ = pA(\psi)$ for some $\psi \in \mathbb{M}$ if and only if ψ is an eigenfunction associated to λ_1^+ .*

Proof. Firstly, we show that λ_1^+ is the least positive eigenvalue of problem (1.1). Indeed, if u is a solution of (1.1) associated to $\lambda \in (0, \lambda_1^+)$, using u as a test function in (2.1), we have

$$\lambda = \frac{\int_I |u''|^p dx}{\int_I m(x)|u|^p dx} \geq \lambda_1^+.$$

This contradicts $\lambda < \lambda_1^+$.

Applying Theorem 1.2 of [31] and Lagrange's multiplier rule, we can easily show the rest results also valid. ■

Furthermore, we also have the following result.

Proposition 3.2. λ_1^+ is simple and the corresponding eigenfunction ψ does not change sign in I .

Proof. Define

$$J_\lambda(u) = A'(u)u - \lambda B'(u)u.$$

It is obvious that u is a solution of (1.1) with $\lambda = \lambda_1^+$ if and only if $J_{\lambda_1^+}(u) = 0$. It is clear that if u is a solution of (1.1) with $\lambda = \lambda_1^+$, then $|u| := \psi$ is also a solution of (1.1) with $\lambda = \lambda_1^+$.

Let u, v be two non-negative solutions of (1.1) with $\lambda = \lambda_1^+$ and put $M(t, x) = \max\{u(x), tv(x)\}$ and $m(t, x) = \min\{u(x), tv(x)\}$. It is obvious that $M, m \in X$. Clearly, $J_{\lambda_1^+}(M) \geq 0$ and $J_{\lambda_1^+}(m) \geq 0$. By simple calculation, we can show

$$J_{\lambda_1^+}(M) + J_{\lambda_1^+}(m) = J_{\lambda_1^+}(u) + t^p J_{\lambda_1^+}(v) = 0.$$

It follows that $J_{\lambda_1^+}(M) = J_{\lambda_1^+}(m) = 0$. Hence, M and m are the solutions of (1.1) with $\lambda = \lambda_1^+$ for all $t \geq 0$. By the continuous embedding of $X \hookrightarrow C^{1,\alpha}(\bar{I})$, we know $u, v \in C^{1,\alpha}(\bar{I})$ and M, m also belong to $C^{1,\alpha}(\bar{I})$ with respect to the second variable.

For any $x_0 \in I$, we divide the rest proof into two cases.

Case 1. $u(x_0) = v(x_0) = 0$.

Clearly, there exists a constant c such that $u(x_0) = cv(x_0)$.

Case 2. $u^2(x_0) + v^2(x_0) \neq 0$.

Without loss of generality, we can assume that $v(x_0) \neq 0$. Set $t_0 = u(x_0)/v(x_0)$. It is clear $+\infty > t_0 \geq 0$. Then we have

$$u(x_0 + t) - u(x_0) \leq M(t_0, x_0 + t) - M(t_0, x_0)$$

and

$$t_0 v(x_0 + t) - t_0 v(x_0) \leq M(t_0, x_0 + t) - M(t_0, x_0).$$

Dividing above two inequalities by $t > 0$ and $t < 0$ and letting t tend to ± 0 , respectively, we obtain

$$\frac{d}{dx}u(x_0) = \frac{\partial}{\partial x}M(t_0, x_0) = t_0 \frac{d}{dx}v(x_0).$$

Furthermore,

$$\frac{d}{dx} \frac{u}{v}(x_0) = \frac{v(x_0) \frac{d}{dx}u(x_0) - u(x_0) \frac{d}{dx}v(x_0)}{v^2(x_0)} = 0.$$

Hence $u(x)/v(x)$ is a constant in I . This completes the proof. ■

Next, we give a version of the Strong Maximum Principle for the p -biharmonic operator with sign-changing weight, which plays an essential role in this paper.

Proposition 3.3. *Eigenfunction ψ corresponding to λ_1^+ is positive in I .*

Proof. Firstly, Proposition 3.2 implies that ψ is non-negative in I . Suppose by contradiction that ψ attains a non-positive minimum over \bar{I} at an interior point. Without loss of generality, we can assume x_0 be the first zero of ψ in I . Set $I_0 := (0, x_0)$.

For any $\phi \in W_0^{1,p}(I_0) \cap W^{2,p}(I_0)$, let $\tilde{\phi}$ be the extension by zero of ϕ on I . It is obvious that $\tilde{\phi} \in X$. By Definition 2.1, we have

$$\int_{I_0} |\psi''|^{p-2} \psi'' \phi'' dx = \int_I |\psi''|^{p-2} \psi'' \tilde{\phi}'' dx = \lambda_1^+ \int_I m(x) |\psi|^{p-2} \psi \tilde{\phi} dx = \lambda_1^+ \int_{I_0} m(x) |\psi|^{p-2} \psi \phi dx. \quad (3.1)$$

Hence the restriction of ψ in I_0 is a non-negative solution of the following problem

$$\begin{cases} \Delta_p^2 u = \lambda_1^+ m(x) |u|^{p-2} u, & x \in (0, x_0), \\ u(0) = u(x_0) = u''(0) = u''(x_0) = 0. \end{cases} \quad (3.2)$$

In fact, the restriction of ψ in I_0 is a classical solution of problem (3.2). Indeed, Proposition 2.1 yields ψ in I is a classical solution of problem (1.1). Hence ψ satisfies $(\varphi_p(\psi''))'' = \lambda_1^+ m(x) \varphi_p(\psi)$ in I_0 .

It remains to show that $\psi''(x_0) = 0$. Let us choose $\phi \in C_0^2(\bar{I}_0) \subset W_0^{1,p}(I_0) \cap W^{2,p}(I_0)$. Substituting in (3.1) and integrating by parts we obtain

$$\psi''(x_0) \phi'(x_0) = 0.$$

Since $\phi'(x_0)$ is arbitrary, it follows from last equality that $\psi''(x_0) = 0$.

Letting $v := \varphi_p(\psi'')$, we divide the rest proof into two cases.

Case 1. $v'(x_0) \neq 0$.

Without loss of generality, we may assume that $v'(x_0) < 0$. Then we can see that $\psi'' > 0$ in $(x_0 - \varepsilon, x_0)$ and $\psi'' < 0$ in $(x_0, x_0 + \varepsilon)$ for some positive constant $\varepsilon > 0$ small enough. Then ψ' is a decreasing function in $(x_0, x_0 + \varepsilon)$. In view of $\psi'(x_0) = 0$, we get $\psi' < 0$ in $(x_0, x_0 + \varepsilon)$, thus ψ is a decreasing function in $(x_0, x_0 + \varepsilon)$. By $\psi(x_0) = 0$, we have $\psi < 0$ in $(x_0, x_0 + \varepsilon)$. We arrive a contradiction.

Case 2. $v'(x_0) = 0$.

For some $x_1 \in (x_0, 1]$, by simple calculation, we show that

$$|v(x_1)| = \left| \int_{x_0}^{x_1} \left(\int_{x_0}^x \lambda_1^+ m(t) \varphi_p(\psi(t)) dt \right) dx \right| \leq \lambda_1^+ |m|_{\infty,0} |\psi|_{\infty,0} |x_1 - x_0|^2,$$

which yields

$$|\psi''|_{\infty,0}^{p-1} \leq \lambda_1^+ |m|_{\infty,0} |\psi''|_{\infty,0} |x_1 - x_0|^2,$$

where $|\cdot|_{\infty,0}$ is the max norm in $[x_0, x_1]$. Taking x_1 be such that $0 < |x_1 - x_0| < (1/\lambda_1^+ |m|_{\infty,0})^{1/2}$, we have $\psi'' = 0$ in $[x_0, x_1]$. Moreover, we also can see $\psi = 0$ in $[x_0, x_1]$. Similarly, we can show

that there exists a number $x_{-1} \in [0, x_0)$ such that $\psi = \psi'' = 0$ in $[x_{-1}, x_0]$.

At the point x_1 , for some positive constant ε_1 small enough, we consider the following three cases.

Case 2.1. $v(x) < 0$ in $(x_1, x_1 + \varepsilon_1)$.

Clearly, $\psi'' < 0$ in $(x_1, x_1 + \varepsilon_1)$. Using the proof similar to *Case 1*, we can show $\psi < 0$ in $(x_1, x_1 + \varepsilon_1)$, which contradicts $\psi \geq 0$ in I .

Case 2.2. $v(x) = 0$ in $(x_1, x_1 + \varepsilon_1)$.

It is obvious that $v'(x_1) = 0$. By the proof similar to *Case 1*, we can obtain there exists a number $x_{21} \in (x_1, 1]$ such that $\psi = \psi'' = 0$ in $[x_1, x_{21}]$.

Case 2.3. $v(x) > 0$ in $(x_1, x_1 + \varepsilon_1)$.

$v(x) > 0$ in $(x_1, x_1 + \varepsilon_1)$ implies that x_1 is a local minimum of v , i.e., $v'(x_1) = 0$. Using the proof similar to *Case 1*, we can obtain there exists a number $x_{22} \in (x_1, 1]$ such that $\psi = \psi'' = 0$ in $[x_1, x_{22}]$.

Set $x_2 := \min\{x_{21}, x_{22}\}$. From above, we have $\psi = \psi'' = 0$ in $[x_1, x_2]$. Similarly, we can show that there exists a number $x_{-2} \in [0, x_1)$ such that $\psi = \psi'' = 0$ in $[x_{-2}, x_0]$. If we repeat the above process again and again, we can show that $\psi \equiv 0$ in I . Contradiction again. \blacksquare

Next, we shall show that λ_1^+ is the unique positive eigenvalue which has positive eigenfunction. The following auxiliary result is needed, which is also interesting in its self.

Lemma 3.1. *For fixed $\lambda > 0$, the eigenvalue problem*

$$\begin{cases} \Delta_p^2 u - \lambda m(x) \varphi_p(u) = \mu \varphi_p(u), & x \in I, \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases} \quad (3.3)$$

has a sequence of eigenvalues

$$-\infty < \mu_1(\lambda) \leq \mu_2(\lambda) \leq \cdots < \mu_k(\lambda) \leq \cdots.$$

Moreover, $\mu_1 = \inf \left\{ \int_I |u''|^p dx - \lambda \int_I m(x) |u|^p dx \mid u \in X, \int_I |u|^p dx = 1 \right\}$.

Proof. Define on X the functional

$$\Phi(u) = \int_I \frac{1}{p} |u''|^p dx - \lambda \int_I \frac{1}{p} m(x) |u|^p dx, \quad \Psi(u) = \int_I \frac{1}{p} |u|^p dx.$$

Set $\mathcal{M} = \{u \in X \mid p\Psi(u) = 1\}$ and

$$\Gamma_k = \{K \subset \mathcal{M} \mid K \text{ is symmetric, compact and } \gamma(K) \geq k\},$$

where $\gamma(K)$ is the genus of K . Then problem (3.3) can be equivalently written as

$$\Phi'(u) = \mu \Psi'(u), \quad u \in \mathcal{M}. \quad (3.4)$$

It is known that (μ, u) solves (3.4) if and only if u is a critical point of Φ with respect to \mathcal{M} . It is well known that \mathcal{M} is a closed symmetric C^1 -submanifold of X with $0 \notin \mathcal{M}$, and $\Phi \in C^1(\mathcal{M}, \mathbb{R})$ is even. It is obvious that

$$\Phi(u) \geq -|\lambda||m|_\infty \quad (3.5)$$

for any $u \in \mathcal{M}$, where $|\cdot|_\infty$ is the max norm in \overline{I} , that is to say Φ is bounded from below.

We claim that Φ satisfies the Palais-Smale condition at any level set c .

Suppose that $\{u_n\} \subset \mathcal{M}$, $|\Phi(u_n)| \leq c$ and $\Phi'(u_n) \rightarrow 0$. Then for any constant $\theta > p$, we get

$$\begin{aligned} c + \|u_n\| &\geq \Phi(u_n) - \frac{1}{\theta} \Phi'(u_n) u_n \\ &\geq \left(\frac{1}{p} - \frac{1}{\theta} \right) \int_I |u_n''|^p dx + \lambda \left(\frac{1}{\theta} - \frac{1}{p} \right) \int_I m(x) |u_n|^p dx \\ &\geq \left(\frac{1}{p} - \frac{1}{\theta} \right) \int_I |u_n''|^p dx - \lambda \left(\frac{1}{p} - \frac{1}{\theta} \right) |m(x)|_\infty. \end{aligned}$$

Hence, $\{\|u_n\|\}$ is bounded. Without loss of generality, we may assume that $u_n \rightharpoonup u$ in \mathcal{M} , then $\Phi'(u_n)(u_n - u) \rightarrow 0$. Hence we have

$$\Phi'(u_n)(u_n - u) = \int_I |u_n''|^{p-2} u_n'' (u_n'' - u'') dx - \lambda \int_I m(x) |u_n|^{p-2} u_n (u_n - u) dx \rightarrow 0.$$

By Hölder's inequality and the compact embedding of $X \hookrightarrow L^p(I)$, we see that

$$0 \leq \left| \int_I m(x) |u_n|^{p-2} u_n (u_n - u) dx \right| \leq |m|_\infty |u_n|_p^{p-1} |u_n - u|_p \rightarrow 0,$$

where $|\cdot|_p$ is the norm on $L^p(I)$. Therefore, we have

$$\int_I |u_n''|^{p-2} u_n'' (u_n'' - u'') dx \rightarrow 0.$$

By Proposition 2.2 (ii), we have $u_n \rightarrow u$.

Now, the result can be obtained by applying Corollary 4.1 of [30]. ■

Using the similar method to prove Proposition 3.2 and 3.3 with obvious changing, we have the following technical result.

Lemma 3.2. $\mu_1(\lambda)$ is simple and the corresponding eigenfunction u_1 is positive in I .

The following technical result is also needed.

Lemma 3.3. (3.3) has a positive solution if and only if $\mu = \mu_1(\lambda)$.

Proof. Define

$$F(u) = \int_I \frac{1}{p} |u''|^p dx + \int_I \frac{1}{p} (c - \lambda m(x)) |u|^p dx, \quad \Psi(u) = \int_I \frac{1}{p} |u|^p dx,$$

where c is a positive constant such that $c - \lambda m(x) \geq 0$ for all $x \in [0, 1]$. Problem (3.3) can be equivalently written as

$$F'(u) = \rho \Psi'(u), \quad u \in \mathcal{M},$$

where $\rho = \mu + c$. Suppose on the contrary that (3.3) with $\mu > \mu_1(\lambda)$ has a positive solution u_2 . Without loss of generality, we assume that $u_1 \leq u_2$ in I .

For any $\phi \in X$ with $\phi \geq 0$, we get

$$\begin{aligned} F'(u_1)\phi &= (\mu_1(\lambda) + c)\Psi'(u_1)\phi \\ &\leq (\mu_1(\lambda) + c) \int_I \varphi_p(u_2)\phi \, dx \\ &= (\mu + c) \int_I \varphi_p(\xi u_2)\phi \, dx \\ &= F'(\xi u_2)\phi, \end{aligned}$$

where $\xi = ((\mu_1(\lambda) + c)/(\mu + c))^{1/(p-1)} < 1$. It follows from Remark 2.2 that $u_1 \leq \xi u_2$. Repeating this argument n times, we get $u_1 \leq \xi^n u_2$. Letting $n \rightarrow +\infty$, we have $u_1 \equiv 0$ in I , which is a contradiction. \blacksquare

Proposition 3.4. *The eigenfunction corresponding to λ_k^+ for any $k > 1$ must change sign in I .*

Proof. Our partial idea arises from [1, 19, 25]. Let

$$S_\lambda = \inf \left\{ \int_I |u''|^p \, dx - \lambda \int_I m(x)|u|^p \, dx \mid u \in X, \int_I |u|^p \, dx = 1 \right\}.$$

(3.5) implies S_λ is bounded below. Using Lemma 3.1, we see $\mu_1(\lambda) = \inf S_\lambda$. Therefore, λ is a principal eigenvalue of (1.1) if and only if $\mu_1(\lambda) = 0$.

For fixed $u \in X$, $\lambda \rightarrow \int_I |u''|^p \, dx - \lambda \int_I m(x)|u|^p \, dx$ is an affine and so concave function. As the infimum of any collection of concave functions is concave, it follows that $\lambda \rightarrow \mu_1(\lambda)$ is a concave function. Also, by considering test functions $\phi_1, \phi_2 \in X$ such that $\int_I m(x)|\phi_1|^p \, dx > 0$ and $\int_I m(x)|\phi_2|^p \, dx < 0$, it is easy to see that $\mu_1(\lambda) \rightarrow -\infty$ as $\lambda \rightarrow +\infty$. Thus, $\lambda \rightarrow \mu_1(\lambda)$ is an increasing function until it attains its maximum, and is a decreasing function thereafter.

Then, as can be seen from the variational characterization of $\mu_1(\lambda)$, $\mu_1(0) > 0$ and so $\lambda \rightarrow \mu_1(\lambda)$ must have exactly one zero. Thus, (1.1) has exactly one principal eigenvalue, $\lambda^+ > 0$, which is simple, and the corresponding eigenfunction that we denoted by $\psi_+(x)$ is positive in I .

We claim that $\lambda^+ = \lambda_1^+$.

Suppose on the contrary that $\lambda^+ \neq \lambda_1^+$. If $\lambda^+ < \lambda_1^+$, then we have

$$\lambda^+ = \frac{\int_I |\psi_+''(x)|^p \, dx}{\int_I m(x)|\psi_+(x)|^p \, dx} \geq \inf_{u \in X, u \neq 0} \frac{\int_I |u''(x)|^p \, dx}{\int_I m(x)|u(x)|^p \, dx} = \lambda_1^+,$$

which is a contradiction.

On the other hand, $\lambda^+ > \lambda_1^+$ is also impossible. Indeed, Lemma 3.1 implies that λ_* is a eigenvalue of (1.1) if and only if there exists a eigenvalue $\mu(\lambda)$ of (3.3) such that $\mu(\lambda) \geq \mu_1(\lambda)$ and $\mu(\lambda_*) = 0$. Hence $\mu(\lambda_1^+) \geq \mu_1(\lambda_1^+) > 0$, that is to say λ_1^+ is not a eigenvalue of (1.1), which is a contradiction again.

Finally, we prove that the eigenfunction corresponding to λ_k^+ for any $k > 1$ must change sign in I . Suppose on the contrary that (1.1) with λ_k^+ has a nonnegative solution u_k^+ for some $k > 1$. Using the similar proof of Proposition 3.3, we can show that u_k^+ is positive. Hence there exists a eigenvalue $\mu(\lambda)$ of (3.3) such that $\mu(\lambda) \geq \mu_1(\lambda)$ and $\mu(\lambda_k^+) = 0$, and (1.1) with $\mu = \mu(\lambda)$ has a positive solution u_k^+ . Lemma 3.3 implies that $\mu(\lambda) = \mu_1(\lambda)$. However, $\mu(\lambda_k^+) = \mu_1(\lambda_k^+) < 0$ which contradicts $\mu(\lambda_k^+) = 0$. \blacksquare

Proposition 3.5. λ_1^+ is isolated; that is, λ_1^+ is the unique eigenvalue in $(0, \delta)$ for some $\delta > \lambda_1^+$.

Proof. Proposition 3.1 has shown that λ_1^+ is left-isolated. Assume by contradiction that there exists a sequence of eigenvalues $\lambda_n \in (\lambda_1^+, \delta)$ which converges to λ_1^+ . Let u_n be the corresponding eigenfunctions. Define

$$\psi_n := \frac{u_n}{\left(\int_I m(x)|u_n|^p dx\right)^{1/p}}.$$

Clearly, ψ_n is bounded in X so there exists a subsequence, denoted again by ψ_n , and $\psi \in X$ such that $\psi_n \rightharpoonup \psi$ in X and $\psi_n \rightarrow \psi$ in $C(\bar{I})$. Since functional A is sequentially weakly lower semi-continuous, we have

$$\int_I |\psi''|^p dx \leq \liminf_{n \rightarrow +\infty} \int_I |\psi_n''|^p dx = \lambda_1^+.$$

On the other hand, $\int_I m(x)|\psi_n|^p dx = 1$ and $\psi_n \rightarrow \psi$ in $C(\bar{I})$ imply that $\int_I m(x)|\psi|^p dx = 1$. Hence $\int_I |\psi''|^p dx = \lambda_1^+$. Then Proposition 3.3 follows that $\psi > 0$ in I . Thus $u_n \geq 0$ for n large enough which contradicts u_n changing-sign in I . \blacksquare

4 The existence of the discrete eigenvalues

From now on, we write $m|_J$ to denote the restriction of m on J for a subset J of I and $Z(u) = \{x \in I \mid u(x) = 0\}$ for simplicity. Moreover, a nodal domain ω of solution u is a component of $I \setminus Z(u)$. In the following, we write $\lambda(p, m, I)$ to denote that λ is dependence on p , m and I , $\lambda(m, I)$ to denote that λ is dependence on m and I .

Clearly, λ_k^+ can be equivalently written as

$$\lambda_k^+ = \inf_{K \in \Lambda_k} \max_{u \in K} \frac{\int_I |u''|^p dx}{\int_I m(x)|u|^p dx}$$

or equivalent to,

$$\frac{1}{\lambda_k^+} = \sup_{K \in \Lambda_k} \min_{u \in K} \frac{\int_I m(x)|u|^p dx}{\int_I |u''|^p dx},$$

which can be written simply,

$$\frac{1}{\lambda_k^+} = \sup_{K \in \Upsilon_k} \min_{u \in K} \int_I m(x)|u|^p dx, \quad (4.1)$$

where $\Upsilon_k = \{K \cap S \mid K \in \Lambda_k\}$ with S is the unit sphere of X . Similarly, λ_1^+ can be written simply,

$$\frac{1}{\lambda_1^+} = \sup_{u \in S} \int_I m(x)|u|^p dx = \int_I m(x)|\psi_+|^p dx. \quad (4.2)$$

In Section 3, we have shown that λ_1^+ is simple, isolated, the unique positive eigenvalue which has an eigenfunction with constant sign and the corresponding eigenfunction $\psi_+ > 0$ in I .

Applying (4.1), (4.2) and the similar method to prove [3, Lemma 1, Lemma 2 and Proposition 1] with obvious changes, we may obtain the following:

Lemma 4.1. (i) Let $m, m' \in M(I)$, $m(x) \leq m'(x)$, then for any k , $\lambda_k^+(m', I) \leq \lambda_k^+(m, I)$.

(ii) Let $(u, \lambda(m, I))$ be a solution pair of (1.1), $m \in M(I)$, then $m|_\omega \in M(\omega)$ for any nodal domain ω of u .

(iii) $\lambda_1^+(m, I)$ verifies the strict monotonicity property with respect to weight m and the domain I , i.e, if $m, m' \in M(I)$, $m(x) \leq m'(x)$ and $m(x) < m'(x)$ in some subset of I of nonzero measure then,

$$\lambda_1^+(m', I) < \lambda_1^+(m, I)$$

and, if J is a strict sub interval of I such that $m|_J \in M(J)$ then,

$$\lambda_1^+(m, I) < \lambda_1^+(m|_J, J).$$

The following lemma play a key role in this section.

Lemma 4.2. The restriction of a nontrivial solution $(u, \lambda(m, I))$ of problem (1.1), on a nodal domain $\omega := (a, b)$, is an eigenfunction of the following problem

$$\begin{cases} \Delta_p^2 u = \lambda m(x) |u|^{p-2} u, & x \in (0, x_0), \\ u(a) = u(b) = u''(a) = u''(b) = 0, \end{cases} \quad (4.3)$$

and we have $\lambda(m, I) = \lambda_1^+(m|_\omega, \omega)$.

Proof. For any $\phi \in W_0^{1,p}(\omega) \cap W^{2,p}(\omega)$, let $\tilde{\phi}$ be the extension by zero of ϕ on I . It is obvious that $\tilde{\phi} \in X$. By Definition 2.1, we have

$$\int_\omega |u''|^{p-2} u'' \phi'' dx = \int_I |u''|^{p-2} u'' \tilde{\phi}'' dx = \lambda \int_I m(x) |u|^{p-2} u \tilde{\phi} dx = \lambda \int_\omega m(x) |u|^{p-2} u \phi dx. \quad (4.4)$$

Hence the restriction of u in ω is a weak solution of problem (4.3) with constant sign. Furthermore, we have $\lambda(m, I) = \lambda_1^+(m|_\omega, I)$.

Similar to Proposition 2.1, the restriction of u in ω is a classical solution of problem (4.3). ■

In the next proposition we give an estimate of the measure of the nodal domains of an eigenfunction u .

Proposition 4.1. Let u be an eigenfunction corresponding to a eigenvalue λ . If $\omega := (a, b)$ is a nodal domain, then

$$|\omega| \geq \left(\frac{1}{\lambda |m|_{\infty, \omega}} \right)^{1/2p},$$

where $|\omega|$ is the Lebesgue measure of the set ω , $|\cdot|_{\infty, \omega}$ is the max norm in ω .

Proof. Lemma 4.2 implies that

$$\int_\omega |u''|^p dx = \lambda \int_\omega m(x) |u|^p dx \leq \lambda |m|_{\infty, \omega} \int_\omega |u|^p dx. \quad (4.5)$$

And Proposition 3.3 implies there exists a point c such that

$$u(c) = \max_{\omega} u(x) > 0.$$

For any t , $x \in \omega$, by simple calculation, we show that

$$u'(t) = \int_c^t u''(\tau) d\tau. \quad (4.6)$$

and

$$u(x) = \int_a^x u'(t) dt. \quad (4.7)$$

By (4.6) and (4.7), we have

$$|u(x)| = \left| \int_a^x \int_c^t u''(\tau) d\tau dt \right| \leq |\omega| \int_a^b |u''(x)| dx. \quad (4.8)$$

Hölder's inequality with (4.8) follows that

$$\int_{\omega} |u(x)|^p dx \leq |\omega|^{p+1} \left(\int_a^b |u''(x)| dx \right)^p \leq |\omega|^{2p} \int_a^b |u''(x)|^p dx. \quad (4.9)$$

Combining (4.5) and (4.9), we have

$$1 \leq \lambda |m|_{\infty, \omega} |\omega|^{2p}, \quad (4.10)$$

which deduces the desired. ■

Remark 4.1. From Proposition 4.1, we also can get an estimate of λ_1^+ from below: $\lambda_1^+ \geq 1/|m|_{\infty}$.

Lemma 4.3. *Each solution $(u, \lambda(m, I))$ of problem (1.1) has a finite number of zeros.*

Proof. Proposition 4.1 implies that u has a finite number of nodal domains. Let $\{I_1, I_2, \dots, I_k\}$ be the nodal domains of u . Put $I_i = (a_i, b_i)$, where $0 \leq a_1 < b_1 \leq a_2 < b_2 \leq \dots < a_k < b_k \leq 1$. It is clear that the restriction of u on $(0, b_1)$ is a nontrivial eigenfunction with constant sign corresponding to $\lambda(m, I)$. Proposition 3.3 yields $u(x) > 0$ for all $x \in (0, b_1)$, so $0 = a_1$. By a similar argument we prove that $b_1 = a_2$, $b_2 = a_3$, \dots , $b_k = 0$, which completes the proof. ■

By the similar method of [3], we also can show the following result.

Proposition 4.2. *There exists a unique real number $c_{2,1} \in I$, for which we have $Z(u) = \{c_{2,1}\}$ for any eigenfunction u corresponding to $\lambda_2^+(m, I)$. For this reason, we shall say that $c_{2,1}$ is the zero of $\lambda_2^+(m, I)$.*

Now, we prove $\lambda_2^+(m, I)$ is simple and the zero $c_{2,1}$ of u is also generalized simple.

Lemma 4.4. *$\lambda_2^+(m, I)$ is simple, hence $\lambda_2^+(m, I) < \lambda_3^+(m, I)$. Moreover, $c_{2,1}$ is the generalized simple zero of u .*

Proof. If $u'(c_{2,1}) \neq 0$, using this fact and the similar method of [3], we can get the conclusion of this lemma. Otherwise, we claim that $v'(c_{2,1}) \neq 0$. Indeed, supposing on the contrary

that $v'(c_{2,1}) = 0$, using the similar proof to Proposition 2.1, we can show $v(c_{2,1}) = 0$. Then via the similar way to Proposition 3.3, we can show $u \equiv 0$ in I . we deduce a contradiction.

Let u and \tilde{u} be two eigenfunctions corresponding to $\lambda_2^+(m, I)$. Similar to the proof of Lemma 6 of [3], we can show that there exist two positive constant α, β such that $u = \alpha\tilde{u}$ in $(0, c_{2,1})$ and $u = \beta\tilde{u}$ in $(c_{2,1}, b)$. By simple calculation, we show that

$$v'(c_{2,1}) = \alpha^{p-1} (\varphi_p(\tilde{u}''(x)))' \big|_{x=c_{2,1}} = \beta^{p-1} (\varphi_p(\tilde{u}''(x)))' \big|_{x=c_{2,1}}. \quad (4.11)$$

It with $v'(c_{2,1}) \neq 0$ follows that $\alpha = \beta$. Noting Definition 2.2, the rest proof is similar to the proof of Lemma 6 of [3]. \blacksquare

The proof of [3, Proposition 3 and Lemma 7] also is valid for our problem, so we can obtain the following two results:

Proposition 4.3. $\lambda_2^+(m, I)$ verifies the strict monotonicity property with respect to the weight m and the domain I .

Lemma 4.5. If any eigenfunction u corresponding to some positive eigenvalue $\lambda(m, I)$ is such that $Z(u) = \{c\}$ for some real number c , then $\lambda(m, I) = \lambda_2^+(m, I)$.

For $k > 2$, we can use the similar recurrence argument of [3] to prove that u_k^+ which is an eigenfunction corresponding to $\lambda_k^+(p)$ has exactly $k - 1$ simple zeros in I .

Proposition 4.4. For any integer k , we have the simple variational formulation,

$$\frac{1}{\lambda_k^+} = \sup_{K \in \mathcal{F}_n} \min_{K \cap S} \int_0^1 m|v|^p dx,$$

where $\mathcal{F}_n = \{K : K \text{ is a } n \text{ dimensional subspace of } X\}$.

Proof. The proof is similar to the proof of Corollary 1 of [3], we omit it here. \blacksquare

Of course, the natural question is that whether or not the eigenvalue functions $\lambda_k^+(p) : (1, +\infty) \rightarrow \mathbb{R}$ is continuous? It is well known the continuity of eigenvalues with respect to p is very important in the study of the global bifurcation phenomena for p -Laplacian or p -biharmonic problems, see [7, 9, 10, 11, 22, 29]. In the following, we will give the confirm answer for this question.

We first show that the principle eigenvalue function $\lambda_1^+ : (1, +\infty) \rightarrow \mathbb{R}$ is continuous.

Proposition 4.5. The eigenvalue function $\lambda_1^+ : (1, +\infty) \rightarrow \mathbb{R}$ is continuous.

Proof. In the following proof, we shall shorten λ_1^+ to λ_1 .

From the variational characterization of $\lambda_1(p)$ it follows that

$$\lambda_1(p) = \sup \left\{ \lambda > 0 \mid \lambda \int_I m(x)|u|^p dx \leq \int_I |u''|^p dx, \text{ for all } u \in C_c^\infty(I) \right\}. \quad (4.12)$$

Let $\{p_j\}_{j=1}^\infty$ be a sequence in $(1, +\infty)$ convergent to $p > 1$. We shall show that

$$\lim_{j \rightarrow +\infty} \lambda_1(p_j) = \lambda_1(p). \quad (4.13)$$

To do this, let $u \in C_c^\infty(I)$. Then, from (4.12),

$$\lambda_1(p_j) \int_I m(x) |u|^{p_j} dx \leq \int_I |u''|^{p_j} dx.$$

On applying the Dominated Convergence Theorem we find

$$\limsup_{j \rightarrow +\infty} \lambda_1(p_j) \int_I m(x) |u|^p dx \leq \int_I |u''|^p dx. \quad (4.14)$$

Relation (4.14), the fact that u is arbitrary and (4.12) yield

$$\limsup_{j \rightarrow +\infty} \lambda_1(p_j) \leq \lambda_1(p).$$

Thus, to prove (4.13) it suffices to show that

$$\liminf_{j \rightarrow +\infty} \lambda_1(p_j) \geq \lambda_1(p). \quad (4.15)$$

Let $\{p_k\}_{k=1}^\infty$ be a subsequence of $\{p_j\}_{j=1}^\infty$ such that $\lim_{k \rightarrow +\infty} \lambda_1(p_k) = \liminf_{j \rightarrow +\infty} \lambda_1(p_j)$.

Let us fix $\varepsilon_0 > 0$ so that $p - \varepsilon_0 > 1$ and for each $0 < \varepsilon < \varepsilon_0$ and $k \in \mathbb{N}$, $p - \varepsilon < p_k < p + \varepsilon$. For $k \in \mathbb{N}$, let us choose $u_k \in W_0^{1,p_k}(I) \cap W^{2,p_k}(I)$ such that

$$\int_0^1 |u_k''|^{p_k} dx = 1 \quad (4.16)$$

and

$$\int_0^1 |u_k''|^{p_k} dx = \lambda_1(p_k) \int_0^1 m(x) |u_k|^{p_k} dx. \quad (4.17)$$

For $0 < \varepsilon < \varepsilon_0$, (4.16) and Hölder's inequality imply that

$$\|u_k''\|_{p-\varepsilon}^{p-\varepsilon} \leq 1. \quad (4.18)$$

This shows that $\{u_k\}_{k=1}^\infty$ is a bounded sequence in $W_0^{1,p-\varepsilon}(I) \cap W^{2,p-\varepsilon}(I)$. Passing to a subsequence if necessary, we can assume that $u_k \rightharpoonup u$ in $W_0^{1,p-\varepsilon}(I) \cap W^{2,p-\varepsilon}(I)$ and hence that $u_k \rightarrow u$ in $C^{1,\beta}(\bar{I})$ with $\beta = 1 - 1/(p - \varepsilon)$ because the embedding of $W_0^{1,p-\varepsilon}(I) \cap W^{2,p-\varepsilon}(I) \hookrightarrow C^{1,\beta}(\bar{I})$ is compact. Thus,

$$|u_k|^{p_k} \rightarrow |u|^p \text{ a.e. } x \in I. \quad (4.19)$$

We note that (4.17) implies that

$$\lambda_1(p_k) \int_0^1 m(x) |u_k|^{p_k} dx = 1 \quad (4.20)$$

for all $k \in \mathbb{N}$. Thus letting $k \rightarrow +\infty$ in (4.20) and using (4.19), we find

$$\liminf_{j \rightarrow +\infty} \lambda_1(p_k) \int_0^1 m(x) |u|^p dx = 1. \quad (4.21)$$

On the other hand, since $u_k \rightharpoonup u$ in $W_0^{1,p-\varepsilon}(I) \cap W^{2,p-\varepsilon}(I)$, from (4.18) we obtain that

$$\|u''\|_{p-\varepsilon}^{p-\varepsilon} \leq \liminf_{k \rightarrow +\infty} \|u_k''\|_{p-\varepsilon}^{p-\varepsilon} \leq 1.$$

Now, letting $\varepsilon \rightarrow 0^+$ we find

$$\|u''\|_p \leq 1. \quad (4.22)$$

Hence $u \in W^{2,p}(I)$. We claim that actually $u \in W_0^{1,p}(I) \cap W^{2,p}(I)$. Indeed, we know that $u \in W^{1,p-\varepsilon}(I)$ for each $0 < \varepsilon < \varepsilon_0$. For $\phi \in C_c^\infty(\mathbb{R})$, it is easy to see that

$$\left| \int_0^1 u \phi' dx \right| \leq \|u'\|_{p-\varepsilon} \|\phi\|_{(p-\varepsilon)'}$$

Then, letting $\varepsilon \rightarrow 0^+$ we obtain that

$$\left| \int_0^1 u \phi' dx \right| \leq \|u'\|_p \|\phi\|_{p'}.$$

Since ϕ is arbitrary, from Proposition IX-18 of [8] we find that $u \in W_0^{1,p}(I) \cap W^{2,p}(I)$, as desired.

Finally, combining (4.21) and (4.22) we obtain

$$\liminf_{j \rightarrow +\infty} \lambda_1(p_k) \int_0^1 m(x) |u|^p dx \geq \int_0^1 |u''|^p dx.$$

This together with the variational characterization of $\lambda_1(p)$ implies (4.15) and hence (4.13). This concludes the proof of the Theorem. \blacksquare

Next, we shall show that eigenvalue functions $\lambda_k^+ : (1, +\infty) \rightarrow \mathbb{R}$, $2 \leq k \in \mathbb{N}$ are continuous.

Proposition 4.6. *For each $2 \leq k \in \mathbb{N}$, the eigenvalue function $\lambda_k^+ : (1, +\infty) \rightarrow \mathbb{R}$ is continuous.*

Proof. Let u_k^+ be an eigenfunction corresponding to $\lambda_k^+(p)$. We have known that u has exactly $k - 1$ simple zeros in I , i.e., there exist $c_{k,1}, \dots, c_{k,k-1} \in I$ such that $u(c_{k,1}) = \dots = u(c_{k,k-1}) = 0$. For convenience, we set $c_{k,0} = 0$, $c_{k,k} = 1$ and $J_i = (c_{k,i-1}, c_{k,i})$ for $i = 1, \dots, k$. Let $\lambda_1^+(p, m|_{J_i}, J_i)$ denote the first positive or negative eigenvalue of the restriction of problem (1.1) on J_i for $i = 1, \dots, k$.

Lemma 4.2 follows that $\lambda_k^+(p) = \lambda_1^+(p, m|_{J_i}, J_i)$ for $i = 1, \dots, k$. By the same method as Proposition 4.5, we can show that $\lambda_1^+(p, m|_{J_i}, J_i)$ is continuous with respect to p for $i = 1, \dots, k$. Therefore, $\lambda_k^+(p)$ is also continuous with respect to p . \blacksquare

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